Wave Propagation on Nonuniform Transmission Lines

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***Abstract*—The problem of wave propagation on nonuniform transmission lines is studied. Equations are presented not only for the reflection coefficient but for the transmission and admittance properties as well. They are valid under the assumption that only one mode of propagation exists on the line and that the properties do not change so rapidly that the fundamental transmission line equations are no longer applicable. Since all equations are derived for arbitrary load conditions, an extremely versatile solution of the problem has been obtained.**

I. A Solution of the Riccati Differential Equation of $\Gamma(x)$

We will study the nonuniform line shown in Fig. 1. The differential equation of the voltage reflection coefficient $\Gamma(x)$ has the form

$$\frac{d\Gamma(x)}{dx} + 2\gamma(x)\cdot\Gamma(x) + g(x)\cdot\Gamma^2(x) = g(x) \tag{1}$$

where we have introduced the factor

$$g(x) = \frac{1}{2}\,\frac{d\ln Y_c(x)}{dx}\,. \tag{2}$$

In these expressions $\gamma(x)$ is the propagation factor and $Y_c(x)$ is the characteristic admittance of the line.

The differential equation is of the Riccati type. A series solution valid at arbitrary load conditions is given by the expression

$$\Gamma(x) = \frac{\phi_1 + \Gamma(0)\cdot\psi_2}{\phi_2 + \Gamma(0)\cdot\psi_1}\cdot\exp\left(-2\int_0^x \gamma(x)\cdot dx\right) \tag{3}$$

where

$$\begin{cases}\phi_1 = K_1 + K_3 + K_5 + \cdots\\ \phi_2 = 1 + K_2 + K_4 + \cdots\\ \psi_1 = Q_1 + Q_3 + Q_5 + \cdots\\ \psi_2 = 1 + Q_2 + Q_4 + \cdots\end{cases}$$

$$\begin{cases}K_1 = \int_0^x f_2(x)\cdot dx\\ K_2 = \int_0^x f_1(x)\cdot K_1\cdot dx\\ K_3 = \int_0^x f_2(x)\cdot K_2\cdot dx\\ K_4 = \int_0^x f_1(x)\cdot K_3\cdot dx\\ \text{etc.}\end{cases}$$

$$\begin{cases}Q_1 = \int_0^x f_1(x)\cdot dx\\ Q_2 = \int_0^x f_2(x)\cdot Q_1\cdot dx\\ Q_3 = \int_0^x f_1(x)\cdot Q_2\cdot dx\\ Q_4 = \int_0^x f_2(x)\cdot Q_3\cdot dx\\ \text{etc.}\end{cases}$$

$$\begin{cases}f_1(x) = g(x)\cdot\exp\left(-2\int_0^x \gamma(x)\cdot dx\right)\\ f_2(x) = g(x)\cdot\exp\left(2\int_0^x \gamma(x)\cdot dx\right).\end{cases}$$

The symbols used in this formula are those given in Fig. 1 and (2).

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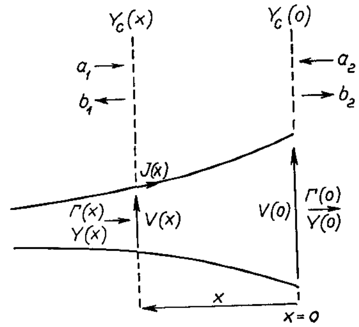

Fig. 1. Nonuniform line with symbols.

Although the solution for $\Gamma(x)$ given by (3) is valid under very general conditions, some restrictions have to be observed. Thus it should be assumed that only one mode of propagation exists on the line. The solution should also be applied with caution to certain types of transmission line tapers which change so rapidly that one might suspect the transmission line equations are no longer valid.

The expression for $\Gamma(x)$ can be given a very simple physical interpretation according to which the terms K_1 and Q_1 correspond to the primary reflections, and K_ν and Q_ν, where $\nu > 1$, to multiple reflections of higher order. Hence one realizes that the calculations can be performed to any desired degree of accuracy by considering a sufficient number of K and Q terms. However, very often the convergence turns out to be so rapid that only a few terms are needed.

The conventional way of solving problems on nonuniform transmission lines is to use an equation that corresponds only to the term K_1 in (3). This solution is obtained immediately from (1) by neglecting the Γ^2 term. This gives a solution which takes into account only the primary reflections and which also assumes matched conditions.

Certain relations exist between the factors included in (3). Thus ϕ_1, ϕ_2, ψ_1, and ψ_2 are interrelated according to the expression

$$\phi_2\psi_2 - \phi_1\psi_1 = 1. \tag{4}$$

Another useful relation can be found between the K and Q terms:

$$K_\nu + Q_\nu = \sum_{n=1}^{\nu-1} (-1)^{n-1}\cdot K_n\cdot Q_{\nu-n}, \qquad \begin{matrix}\nu = 2, 4, 6, 8, \cdots\\ n = 1, 2, 3, \cdots, (\nu-1).\end{matrix} \tag{5}$$

II. The Scattering Coefficients of the Nonuniform Line

Studies on nonuniform lines reported in the literature seem to have been almost exclusively dealing with the reflection coefficient, whereas the equally important transmission properties have received little, if any, attention. The method introduced here, however, lends itself excellently to determining the latter properties as well.

Being able to calculate both reflection and transmission properties of the nonuniform line, we can also determine its scattering coefficients. By using the notations in Fig. 1 and (3) we obtain the following expressions:

$$\begin{cases}S_{11} = \dfrac{b_1}{a_1} = \dfrac{\phi_1}{\phi_2}\cdot\exp\left(-2\int_0^x \gamma(x)\cdot dx\right)\\ S_{22} = \dfrac{b_2}{a_2} = -\dfrac{\psi_1}{\phi_2}\\ S_{21} = \dfrac{b_1}{a_2} = \dfrac{1}{\phi_2}\cdot\exp\left(-\int_0^x \gamma(x)\cdot dx\right)\\ S_{12} = \dfrac{b_2}{a_1} = \dfrac{1}{\phi_2}\cdot\exp\left(-\int_0^x \gamma(x)\cdot dx\right).\end{cases} \tag{6}$$

Here a_i and b_i denote incident and reflected voltage waves at the two ports as shown in Fig. 1.

III. The Admittance of the Nonuniform Line

Sometimes it is advantageous to determine the admittance $Y(x)$ instead of the reflection coefficient of the nonuniform line. This is especially true when the series impedance $Z_s(x)$ and the shunt admit-

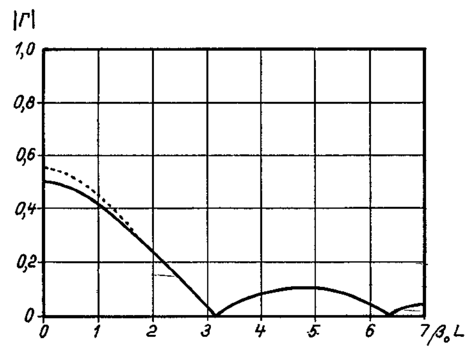

Fig. 2. $|\Gamma|$ versus $\beta_0 L$ at an impedance ratio of 3:1.

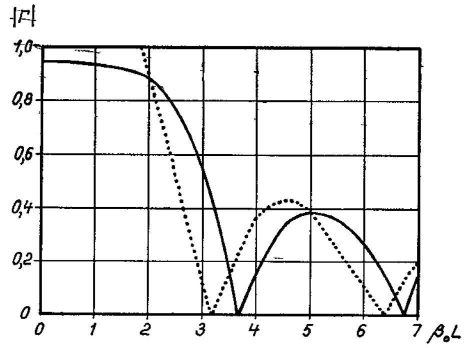

Fig. 3. $|\Gamma|$ versus $\beta_0 L$ at an impedance ratio of 54:1.

tance $Y_p(x)$ of the line are small, as is often the case at low frequencies. Also in this case we have to solve a Riccati equation, and the solution is given by (3) if the exponential term is omitted and the symbols are exchanged according to the scheme

$$\begin{cases}\Gamma(x) \rightarrow Y(x) \\ \Gamma(0) \rightarrow Y(0) \\ f_1(x) \rightarrow Z_s(x) \\ f_2(x) \rightarrow Y_p(x). \end{cases} \tag{7}$$

The solution for $Y(x)$ has turned out to be particularly useful, for example in calculations of the propagation of plane waves in lossy, stratified media at low frequencies.

IV. A Numerical Example

In order to give an idea of the usefulness of (3) in a practical case, the magnitude of the reflection coefficient $\Gamma(x)$ of a matched exponential line has been computed for two different impedance ratios. This particular line has been chosen since it is also possible to derive an exact expression for its reflection coefficient which can be compared with (3). The characteristic admittance of the exponential line is supposed to vary as $Y_c(x) = Y_0 \cdot \exp(2\delta x)$, while its propagation factor is $\gamma = j\beta_0$. The two values of δL that have been used for δx (δ is a constant and L is the length of the line) are $\delta L = 0.55$ and $\delta L = 2$, corresponding to impedance ratios of 3:1 and 54:1, respectively. (Since this is only an illustrative example, no attention is payed here to the physical realizability of such lines.) The highest order term used is K_5. At the lowest values of $\beta_0 L$, the equation for $Y(x)$ has been used instead of (3) to obtain better convergence.

The results, which are shown in Figs. 2 and 3, are rather striking. The curves one gets with the method presented here (solid curves) coincide within drawing accuracy with the exact ones for all values of $\beta_0 L$ at both impedance ratios. The curves obtained when the problem is solved in the conventional way (dotted curves), on the other hand, show serious disagreement particularly for the high impedance ratio and at low values of $\beta_0 L$.

V. Conclusion

In this short paper equations are presented for the electrical properties of a nonuniform line. They are given in series form and are valid also for lossy lines connected to arbitrary loads. The equations may be applied to all kinds of single-mode transmission lines; for instance, coaxial lines, strip lines, and waveguides. As a consequence they can be utilized in the design of many microwave components containing nonuniform line sections, like resonators, filters, tapered transitions, etc. In cases where one has the choice, nonuniform line sections often have advantages over uniform ones. The usefulness of the equations derived here is, however, not limited to transmission lines only. Due to the analogy between the free propagation of plane waves in a medium and waves on transmission lines, the results may also be used in the design of certain types of absorbing materials or in the study of propagation of plane waves in a stratified atmosphere, to take only two examples.

The well-known fact that the Riccati equation can be transformed by a simple mathematical operation into a one-dimensional wave equation indicates that the equations may be applied in other fields of physics as well. Thus, for example, it may well be expected that the solution described in this short paper could be used with benefit in such fields as acoustics, optics, and quantum mechanics.

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The Lowest Order Mode and the Quasi-TEM Mode in a Ferrite-Filled Coaxial Line or Resonator

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***Abstract*—The field distribution of the mode in a ferrite-filled coaxial cavity, which converges towards the TEM mode in the isotropic case, is discussed.**

During the last few years there has been a discussion between M. M. Weiner and M. E. Brodwin and D. A. Miller about the "lowest order mode" and an approximate theory for this mode, called the quasi-TEM mode, in a ferrite-filled coaxial line [1]–[3]. The author has studied the behavior of all modes in a ferrite-filled coaxial cavity [4], [5] and would like to give some detailed results for the "lowest order mode" and the correct conditions for approximating it by the Suhl and Walker approximation [7] of a quasi-TEM mode.

Basically, there are three different kinds of modes in a ferrite-filled cavity.

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